A novel regularity criterion for the three-dimensional Navier-Stokes equations based on finitely many observations.

Abstract: The Navier-Stokes equations (NSE) is a system of two partial differential equations governing fluid flow. It describes fluid motions under quite general conditions, from laminar to turbulent and on scales ranging from below a millimeter to astronomical lengths. Consequently, these equations have wide ranging applications in areas such as aeronautical sciences, meteorology, petroleum industry, plasma physics, and more recently, in biophysical fluid dynamics. The analytical challenges concerning these equations, particularly in three spatial dimensions, are manifested among others in the problem of long-term existence of *regular* solutions, uniqueness of *weak solutions,* and the study of qualitative behavior of solutions for large times, not to mention issues related to stability and convergence of various numerical approximation schemes. Lying at the very heart of the matter is the question of whether the solutions to these equations develop singularities as they evolve in time. It is possible that the equations generate solutions that produce singularities spontaneously after finite time. If this happens, then subsequent evolution may be non-unique, violating the basic tenets of Newtonian determinism for this model. Finite time singularities also indicate that the equations are generating structures on arbitrarily small scales, contradicting the separation-of- scales assumption in many microscopic models of hydrodynamics. Thus, the questions of long-time existence, uniqueness and regularity (i.e., lack of singularity) are intimately connected to the efficacy of these equations as models for fluid turbulence. We present a novel regularity criterion for the 3D Navier-Stokes equations (NSE) based on finitely many *modal, nodal or volume element observations* of the velocity field. The regularity criterion we propose is intimately connected to the notion of *determining functionals* (modes, nodes and volume elements) which in turn is related to the notion of Landau-Lefschetz degrees of freedom for turbulent flows. To the best of our knowledge, all existing regularity criteria require knowing the solution of the 3D NSE almost everywhere in space. Our regularity criterion is fundamentally different from any preexisting regularity criterion as it is based on finitely many observations (modes, nodes and volume elements). We further prove that the regularity criterion we propose is both a necessary and sufficient condition for regularity. Thus, our result can be viewed as a natural generalization of the notion of determining modes, nodes and volume elements as well as the asymptotic tracking property of a data assimilation algorithm for the two-dimensional NSE to the three-dimensional setting.